Horton's Law of Stream Order Numbers and a Temperature-Analog in River Nets

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Abstract. Statements of Horton's law in Horton, Strahler, and consistent stream ordering systems are presented. A topological characterization of a 'Horton net' is given. It is then shown how the analog of 'temperature' can be defined in microcanonical ensembles of stream nets, based on an earlier representation of river nets by graphs (bifurcating arborescences). (Key words: Rivers; geomorphology)

INTRODUCTION

Horton's [1945] 'law of stream order numbers' has been the subject of many investigations. It represents a statistical relationship for the numbers of rivers or river segments of various 'orders' present in a drainage basin. Originally, it was formulated for a stream-ordering system due to *Horton* [1945], but later it was used for other (notably the Strahler) types of stream-ordering systems.

In effect, if Horton's law is valid for a river system and its parts with the same bifurcation ratio, this implies a very special characterization of the (topological) 'graph' representing the river system. The latter must form what we call a 'Horton net.' Thus, it appears as most sensible to study the mathematical properties of such graphs.

It then becomes obvious that any valid characterization of a 'Horton net' represents an equivalent statement of Horton's law. In this fashion, Horton's law can easily be stated in terms of various river ordering systems, such as in the Horton, *Strahler* [1957], *Woldenberg* [1966], or consistent [*Scheidegger,* 1965] systems. Moreover, since the connection with graph theory is clear, the combinatorial topology of such graphs presents the obvious clue in searching for a connection between stochastic river-net theory and statistical principles of theoretical physics. Thus, it is shown how a canonical ensemble of graphs can be defined and therewith an analog of the 'temperature' of gas dynamics can be obtained.

HORTON'S LAW OF STREAM ORDER NUMBERS IN HORTON AND STRAHLER ORDERS

Horton's law of stream numbers states a topological relationship for rivers making up a drainage network. The concept of 'stream order' in a drainage network now widely used in the United States was introduced by *Horton* [1945]: First-order streams are those that have no tributaries, second-order streams those that have as tributaries only streams of first order, third-order streams those that have as tributaries only streams of second or lower order, etc. Thus, when two streams of equal order meet, the order of the combination is increased by 1. However, Horton thought that the 'main stream' should be denoted by the same order number from the headwaters to its mouth, and so he renumbered at every junction of two rivers of order n, one to the order $n + 1$. This procedure is repeated at every junction of equal-order streams. Conversely, *Strahler* [1957] assigned order numbers only to stream segments: If two segments of order n meet, they form a stream segment of order $n + 1$.

In Horton's as well as Strahler's ordering systems, junctions of nth order streams with lower order streams are simply ignored.

Horton's law of stream numbers can now be stated as follows: If we denote by n_i^H the number of (Horton) rivers of order i in a given network, then the numbers n_1^H, $n_2^H \cdots$ form a geometric sequence

$$n_{i+1}^H = \beta n_i^H \tag{1}$$

One often calls the inverse of β, i.e., $1/\beta$, the 'bifurcation ratio' of the river net.

Horton's law of stream numbers is, in effect, a topological characterization of a network. Such networks may conveniently be referred to as 'Horton nets.'

We thus define a Horton net as a net in which (on the average) $1/\beta$ rivers of order n combine to form a river of order $n + 1$. In such a network, Horton's law of stream numbers is automatically satisfied. It may be a matter of curiosity that the converse is not necessarily true: It is, in fact, possible to construct a river basin of order n that satisfies Horton's law of stream numbers, but that is not a Horton net as defined above. However, in such a basin, the (largest) subbasins (of a given order smaller than n) no longer satisfy Horton's law with the same bifurcation ratio as the original basin. Hence, it seems reasonable to restrict 'Hortonian' networks to those that not only obey Horton's law themselves, but in which also the (largest) subbasins of all orders smaller than n satisfy this law with the same bifurcation ratio. This corresponds to the definition given above.

Horton's law as stated above refers to 'Horton orders.' However, it is easy to show that it could also have been stated in terms of Strahler orders. Let us denote the number of river segments (in the Strahler sense) of order i by n_i^s. Horton's law then maintains that the numbers $n_1^s, n_2^s \cdots$ in a river network form a geometric sequence

$$n_{i+1}^s = \beta n_i^s \tag{2}$$

If the above relationship is satisfied in the Strahler sense, then it is also satisfied in the Horton sense. We have

$$n_i^H = n_i^s - n_{i+1}^s \tag{3}$$

since exactly $n_i^s + 1$ of the n_i^s Strahler segments have been renumbered to obtain Horton rivers. Thus

$$n_i^H/n_{i+1}^H = (n_i^s - n_{n+1}^s)/(n_{i+1}^s - n_{i+2}^s)$$

With

$$n_{i+1}^s = \beta n_i^s$$

we get

$$\frac{n_i^H}{n_{i+1}^H} = \frac{n_i^s - \beta n_i^s}{\beta n_i^s - \beta^2 n_i^s}$$

$$= \frac{n_i^s(1-\beta)}{\beta n_i^s - \beta^2 n_i^s} = \frac{n_i^s(1-\beta)}{\beta n_i^s(1-\beta)} = \frac{1}{\beta}$$

so that, *in a Horton net*

$$n_i^H/n_{i+1}^H = n_i^s/n_{i+1}^s = 1/\beta \tag{4}$$

It should be noted, however, that the deduction of the above relationship presupposes that, if the bifurcation ratio from i + first-order streams to ith order streams is under consideration, the network also contains i + second-order streams.

HORTON'S LAW IN CONSISTENT STREAM ORDERS

Horton's law was stated above for Strahler-type stream orders. Such Strahler orders do not take account of river segments of different orders that might form a 'junction'; the lower-order stream simply gets lost. The writer [*Scheidegger*, 1965] has, therefore, suggested the introduction of 'consistent' stream orders, defined by a logarithmic composition law: if segments of orders M and N join, the result is a segment of order X with

$$X = \log_2 (2^N + 2^M) \tag{5}$$

or

$$2^X = 2^N + 2^M \tag{6}$$

Instead of the orders, one can therefore use the associated integers. In a completely 'regular' stream net, i.e., in one where each river joins one of the same order (this corresponds to a 'Horton net' with bifurcation ratio 2), consistent orders and Strahler orders are the same. Often, it is useful to use the 'associated integers' 2^M to designate the order of a stream rather than the order M.

It may now be of some interest to investigate how Horton's law of stream numbers is expressed in consistent rather than in Strahler or Horton orders.

Thus, we assume that the net forms a 'Horton net.' Such networks can be visualized easily if $1/\beta$ is an integer: Then exactly $1/\beta$ first-order streams will form one second-order (Strahler order) stream, $1/\beta$ (Strahler) second-order streams will form one third-order (Strahler) stream, etc. As indicated, these orders are Strahler orders, but the consistent orders can easily be calculated. Let us assume that, in Strahler order n, the consistent order is N, and the associated integer, therefore, is $\nu = 2^N$. In the Horton net, then, there are exactly $1/\beta$ nth-order (Strahler) streams that form each n + first-order (Strahler) stream. Thus, by induction, starting with $n = 1$, $\nu = 2$, we have that for any river segment of Strahler order n, the (maximum) associated consistent integer ν is

$$\nu = 2(1/\beta)^{n-1} \tag{7}$$

and the consistent order N

$$N(n) = \log_2 2(1/\beta)^{n-1}$$
$$= 1 + (n - 1) \log_2 (1/\beta) \tag{8}$$

As it should be in conformity with earlier remarks, we have $N(n) = n$ if $1/\beta = 2$. By analytic continuation, the above formula will also be assumed to be valid for noninteger bifurcation ratios. Of course, a 'Horton net' cannot really have a noninteger bifurcation ratio; every time there must be, say, 3 or else 4 nth-order streams that form a n + first-order stream, but on the average, in a statistical ensemble of river nets, this can be envisaged to occur.

Woldenberg [*Warntz and Woldenberg*, 1967] introduced a stream order that might well be called 'natural' stream order by using, instead of 2, the bifurcation ratio $1/\beta$ as the base of the logarithm and as the base of the exponent in the composition law of stream orders. However, Woldenberg did not note that such natural stream orders, designated here by $\mathfrak{N}$, cannot, of course, be defined for networks in which Horton's law of stream numbers is not, at least on the average, valid.

One can again introduce associated quantities with the natural stream orders. One would write

$$\mathfrak{n} = (1/\beta)^{\mathfrak{N}} \tag{9}$$

and then have for the analogous meaning as for the quantities 2 above: $\mathfrak{n}$ is $1/\beta$ times the number of first-order streams making up the network to the order under consideration (i.e., after each 'cycle' is complete). A related quantity

$$w = (1/\beta)^{\mathfrak{N}-1} \tag{10}$$

has been called 'absolute' stream order by *Woldenberg* [1966]. It is again stated, however, that 'absolute' orders as well as natural orders of Woldenberg can only be defined for Horton nets, in contrast to 'consistent' orders, which can always be defined.

We now proceed to formulate a convenient characterization of a net as a Horton net in terms of consistent orders.

In natural orders, we know that the number of tributaries ΔS along a main stem (the order of the 'main stem' must always be equal to or larger than that of the 'tributaries') for a change of orders $\Delta\mathfrak{N} = 1$ is exactly $1/\beta - 1$, or

$$\Delta S = (1/\beta - 1)\ \Delta\mathfrak{N} \tag{11}$$

However, natural and consistent orders (in a Horton net) are linearly related, so that (see equation 8)

$$\Delta N = \Delta\mathfrak{N} \log_2 1/\beta \tag{12}$$

Hence,

$$\Delta S = (1/\beta - 1)\ \Delta N/[\log_2 (1/\beta)] \tag{13}$$

The last formula yields a practical means for testing whether a given river net is a Horton net or not. In the limit, we see that, using consistent orders (these are defined for *any* river net, not only Horton nets), we can write

$$dS/dN = \text{const.}$$
$$= (1/\beta - 1)/[\log_2 (1/\beta)] \tag{14}$$

which must be satisfied at least in the mean if the net is to be a Horton net. The bifurcation ratio $1/\beta$ can then be calculated from the constant.

DEFINITION OF A 'TEMPERATURE' ANALOG IN A STREAM NET

It is well known that, in many fluctuating systems, it is possible to define a temperature analog [*Scheidegger*, 1961]. In the case that the fluctuating quantity is the energy, the temperature is simply proportional to the mean value of the energy in each component 'cell' of the system; in the theory of ideal gases, the cells are represented by single degrees of freedom. The existence of a nonnegative quantity $\mathfrak{A}$, which is a constant of the motion for the (isolated) system as a whole, is always characteristic for the validity of a temperature analog. This quantity $\mathfrak{A}$ may be the energy, mass, or some other quantity.

If the system then consists of component systems ('cells') between which there is a weak interaction of fairly general type, then the quantity $\mathfrak{A}$ in question is canonically distributed over the 'cells', and a temperature analog holds.

The statistical distribution of the quantity in question over the cells can also be obtained by regarding an ensemble of 'whole systems' that all have the same value for the quantity $\mathfrak{A}$ but differ in detail. Focusing attention on a particular cell while the whole system assumes all possible configurations with equal probability (microcanonical ensemble) will again produce the canonical distribution for the quantity $\mathfrak{A}$ in that cell, and therewith a temperature analog.

The question now is whether an 'ensemble theory' with an attendant temperature analog can be set up for river nets. As is evident, one must define (1) a nonnegative constant of the motion $\mathfrak{A}$ for a whole river net, (2) a microcanonical ensemble of river nets, and (3) the notion of a cell.

The writer [*Scheidegger*, 1967] has already studied the statistics of river nets as a whole by comparing them with mathematical graphs of a special type, that is, with bifurcating arborescences. It was shown, then, that a microcanonical ensemble can be set up by considering all possible graphs with a given number of pendant vertices. Thus, the number of pendant vertices could be considered as the quantity $\mathfrak{A}$ in a temperature-analog scheme for rivers. However, if one wants to define subgraphs (playing the role of cells), the number of pendant vertices in the total graph is a very inconvenient quantity, since the subgraphs do not directly contribute to it. A much more convenient quantity is the number of junctions, which for N pendant vertices is $N - 1$. We take the latter number as our quantity $\mathfrak{A}$.

We thus have defined a microcanonical ensemble: this is the ensemble of all possible graphs (bifurcating arborescences) with a given number $\mathfrak{A}$ of junctions. In an earlier paper [*Scheidegger*, 1967] it had been shown that the expectation value of the bifurcation ratio in such a microcanonical ensemble is not only constant but is even numerically equal to that observed in mature rivers of the United States.

It remains to define the notion of a 'cell.' These cells must contain, under equilibrium conditions, an equal share of the quantity $\mathfrak{A}$, i.e., an equal number of junctions. If we assume that a *Horton net* represents the equilibrium configuration, we can use the characterization of such a net in terms of consistent stream orders to define 'cells': A cell is a linear distance along a main stream (stream into which only tributaries of lower order flow) over which the consistent stream order changes by a constant amount (say, by $+1$). The number $\mathfrak{A}_i$ of junctions in every cell (i) in a Horton net will be exactly constant; in other nets, it will not be constant. Thus, we can define a temperature-analog T_i for cell i by the relation

$$kT_i = \mathfrak{A}_i \tag{15}$$

where k is some proportionality constant akin to Boltzmann's constant in gas theory.

In others words, the temperature-analog in a river net is simply proportional to the local bifurcation ratio less 1 for a change in (consistent) orders by some constant quantity.

It is obviously tempting to calculate 'local' bifurcation ratios using Strahler rather than consistent stream orders, as the Strahler orders are more easily determinable. It is clear that the general condition for a Horton net can be just as easily expressed in Strahler orders as in consistent orders, inasmuch as, *if* a river net is a Horton net, Strahler orders are strictly proportional to consistent orders. For Horton nets there is therefore a unique correspondence between Strahler bifurcation ratios and the 'temperature.'

The matter is not so simple if the net is not a Horton net, and one might well ask what the relationship is between the local Strahler bifurcation ratio and the consistent local 'temperature' of the net. A simple analysis of a non-Hortonian network makes it evident, however, that such a relationship cannot be defined uniquely. Thus, let us assume that the local Strahler-bifurcation ratio is $1/\beta$, so that $1/\beta$ streams form a river of order n. Two of these $1/\beta$ streams must be of (Strahler) order $(n - 1)$; the others must be of lower order. Since this 'lower order,' however, is not specified, it is immediately obvious that no unique consistent orders can be assigned to them. Thus, there is no unique relation in a non-Hortonian river net between the 'temperature'-analog and local Strahler bifurcation ratios.

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REFERENCES

Horton, R. E., Erosional development of streams and their drainage basins; hydrophysical approach to quantitative morphology, *Geol. Soc. Am. Bull. 56,* 275–370, 1945.

Scheidegger, A. E., On the statistical properties of some transport equations, *Can. J. Phys. 39,* 1573–1582, 1961.

Scheidegger, A. E., The algebra of stream order numbers, *U. S. Geol. Surv. Prof. Paper 525B,* B187–B189, 1965.

Scheidegger, A. E., Random graph patterns of drainage basins, Paper 157, *Int. Assoc. Scient. Hydrol.* U.G.G.I. General Assembly, Rome, 1957.

Strahler, A. N., Quantitative analysis of watershed geomorphology, *Trans. Am. Geophys. Union, 38,* 913–920, 1957.

Warntz, W., and M. Woldenberg, Concepts and applications, spatial orders, *Harvard Papers in Theoretical Geography, Geography and the Properties of Surfaces Series* No. 1, 187 pp., 1967.

Woldenberg, M. J., Horton's laws justified in terms of allometric growth and steady state in open systems, *Geol. Soc. Am. Bull. 77,* 431–434, 1966.

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